OFFICE OF COMMUNICATIONS

Award of spectrum: 1452 to 1492 MHz

Notes on determination of licence fees payable by bidders in relation to the award of the 1452 to 1492 MHz spectrum band: Schedule 6 to the Wireless Telegraphy (Licence Award) Regulations 2008

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Contents

Section 1

Introduction

1.1 The Wireless Telegraphy (Licence Award) Regulations 2008 (the "Regulations") set out the process that Ofcom will use for the award of wireless telegraphy licences for the use of the 1452 to 1492 MHz spectrum band (the "band").

1.2 One of the features of the award process is that the licence fees payable to Ofcom by winning bidders are determined by Ofcom in accordance with rules set out in the Regulations.

1.3 The purpose of this document is to provide further information about the determination of these licence fees. Ofcom expects that this will be of interest mainly to prospective applicants for licences for the use of the band.

Document structure

1.4 Section 2 provides a summary of the process set out in Part 5 of the Regulations.

1.5 Section 3 provides a summary of the requirements for the calculation of "winning prices" in Schedule 6 to the Regulations.

1.6 Section 4 explains in mathematical terms why, for any particular set of valid auction stage bids and winning combination of auction stage bids, there will be a unique set of winning prices that satisfies the requirements in Schedule 6.

1.7 Section 5 gives contact details for further information.

Section 2

The award process and determination of licence fees

2.1 The award process set out in Part 5 of the Regulations involves two stages: the auction stage and the grant stage.

2.2 During the auction stage there are one or more rounds during which primary bids may be made, and then a further round for making supplementary bids.

2.3 When making a primary bid during a primary bid round, a bidder specifies the selection of lots in the band it wishes to be included in a licence. The price for each lot in the band is set by Ofcom for that primary bid round. As the auction progresses, prices for lots either remain the same or else are increased by Ofcom, in accordance with regulation 19.

2.4 After the last primary bid round, there is a supplementary bids round. Bidders wishing to make supplementary bids specify the selections of lots in the band they wish to be included in a licence. The price that bidders are willing to pay for each selection of lots is specified by the bidder, subject to certain restrictions which are set out in regulation 26.

2.5 At the end of the auction stage Ofcom determines (in accordance with regulation 41) the winning auction stage bids and the winning combination of auction stage bids.

2.6 Regulation 43 then requires Ofcom to determine an amount, in accordance with Schedule 6, for each winning auction stage bid which is payable by the relevant winning bidder. This amount is defined in the Regulations as the “winning price”.

2.7 The licence fee payable by a winning bidder to Ofcom is the winning price for its winning auction stage bid.

Section 3

Calculation by Ofcom of winning prices

3.1 The following section provides a summary of the requirements for the calculation of winning prices in Schedule 6 to the Regulations.

3.2 Under Schedule 6, Ofcom will determine a winning price for each winning auction stage bid by the imposition of requirements that the winning prices must satisfy. There are four requirements and they are set out in Schedule 6.

3.3 The first requirement in Schedule 6 is that the winning price for a winning auction stage bid must be no less than the round prices for the first primary bid round for the lots selected in the bid, and must be no greater than the amount of the bid.

3.4 The second and third requirements in Schedule 6 apply to the winning prices for all of the winning auction stage bids taken together, rather than individually. The effect of these two requirements is that the total of all of the winning prices taken together (with one winning price for each winning auction stage bid) must be the lowest possible total such that, if the winning auction stage bids had been for the amount of the corresponding winning prices, and all other valid auction stage bids made by the winning bidders had been similarly reduced, the winning combination of auction stage bids would still have been the combination (or one of the combinations) of valid auction stage bids with the highest value.

3.5 The fourth requirement again involves looking at the winning prices for all of the winning auction stage bids taken together. In broad terms, the effect of this requirement is that where there is more than one set of prices that satisfies the first, second and third requirements, the winning prices must be the set of prices with the lowest value of what the Regulations call the "opportunity cost variance". The opportunity cost variance is a way of measuring the difference between the opportunity cost of winning auction stage bids and the amount of the winning prices for those bids.

3.6 It is a feature of these four requirements that, for any particular set of valid auction stage bids and winning combination of auction stage bids, there will be only one set of winning prices (comprising one winning price for each winning bid) that satisfies those requirements.

3.7 Section 4 explains, in mathematical terms, why this is necessarily the case.

3.8 To find the winning prices that satisfy the requirements in Schedule 6 it is necessary to apply mathematical concepts from a branch of mathematics called 'convex optimisation' – the branch of mathematics concerned with finding optimal solutions to these types of problem.

3.9 There are a number of different methods (or 'algorithms') that may be used to apply those mathematical concepts to identify the winning prices that satisfy the four requirements set out in Schedule 6. The Regulations do not prescribe which of these algorithms Ofcom must use. This is because, as there can only be one set of prices that satisfies the four requirements in Schedule 6, each of these different algorithms will produce the same answer.

3.10 The possible algorithms all have in common the fact that they involve a very large number of individual calculations, such that in practice it is necessary to apply the algorithm through mathematical software.

3.11 Ofcom has commissioned software to apply one of the algorithms that may be used to determine winning prices that meet the requirements in Schedule 6.

Section 4

Uniqueness of winning prices

4.1 The uniqueness of the winning price for each winning bid that meets the requirements set out in Schedule 6 follows directly from a standard result in convex optimisation. This result states that there is a unique solution to the problem of minimising a strictly convex function over a closed bounded convex region. In the case of winning price determination, the Opportunity Cost Variance is a strictly convex function of prices, and the first three requirements in the Schedule define a closed bounded convex set of prices over which the Opportunity Cost Variance is to be minimised. The result then follows directly.

4.2 For the purposes of completeness however, in the remainder of this note we prove that winning prices exist, and are unique, from first principles. We use the following notation:

- let *i* index over bidders;
- let *j* index over the valid (auction stage) bids made by each bidder;
- let *k* index over the available lots;
- let b_{ij} denote the amount (in pounds) of the *j*th bid by bidder *i*;
- let q_{ijk} denote the inclusion, $q_{ijk} = 1$, or exclusion, $q_{ijk} = 0$, of lot *k* in the selection of lots of the *j*th bid by bidder *i*;
- let x_{ij} denote the inclusion, $x_{ij} = 1$, or exclusion, $x_{ij} = 0$, of the *j*th bid by bidder *i* in the combination of bids ***x*** (note that x_{ij} can only equal 0 or 1).

4.3 A combination of bids, ***x***, is then a valid combination of bids if and only if it satisfies both of the following two conditions:

Condition 1

$\sum_j x_{ij} \le 1$ for all bidders *i*

(i.e. a valid combination of bids includes at most one bid from each bidder).

Condition 2

$\sum_i \sum_j x_{ij} . q_{ijk} \le 1$ for all lots *k*

(i.e. for each and every lot, that lot is selected in at most one bid in each valid combination of bids).

4.4 Now let ***x**** denote the **winning combination of bids**. Then ***x**** must be a valid combination of bids and must also satisfy the following condition:

Condition 3

$\sum_i \sum_j x_{ij}^* . b_{ij} \geq \sum_i \sum_j x_{ij} . b_{ij}$ for all valid combinations of bids ***x***

(i.e. the winning combination of bids is one of the valid combinations of bids for which the total amount (in pounds) of the bids included in the combination is greatest).[1]

4.5 To simplify later notation:

- let b_i^* denote the amount of the winning bid of bidder *i* (if any), with $b_i^* = 0$ if bidder *i* is not a winning bidder;
- let r_i^* denote the total of the round prices in the first primary bid round for the selection of lots in the winning bid of bidder *i* (if any), with $r_i^* = 0$ if bidder *i* is not a winning bidder;
- let c_i^* denote the opportunity cost of the winning bid of bidder *i* (if any), with $c_i^* = 0$ if bidder *i* is not a winning bidder;
- let p_i denote the price for the winning bid of bidder *i* (if any) in the vector of prices for winning bids ***p*** (with $p_i = 0$ if bidder *i* is not a winning bidder).

4.6 Now let ***p**** denote the vector of **winning prices** for winning bids i.e. p_i^* is the winning price for the winning bid of bidder *i* (if any), with $p_i^* = 0$ if bidder *i* is not a winning bidder.

4.7 The requirements of Schedule 6 can then be expressed mathematically as follows:

First requirement

$r_i^* \leq p_i^* \leq b_i^*$ for all bidders *i*.

Second requirement

$\sum_i \sum_j x_{ij}^* . \left(b_{ij} - \left(b_i^* - p_i^*\right)\right) \geq \sum_i \sum_j x_{ij} . \left(b_{ij} - \left(b_i^* - p_i^*\right)\right)$

for all valid combinations of bids ***x***.

Third requirement

$\sum_i p_i^* \leq \sum_i p_i$

for all vectors of prices for winning bids ***p*** that satisfy the first and second requirements.

[1] NB Condition 3 is necessary but not always sufficient to identify the winning combination of bids. In some cases there may be a number of valid combinations of bids that all satisfy Condition 3. In this case we use the order of precedence set out in the Regulations to determine which of these combinations of bids is the winning combination.

Fourth requirement

$$OCV(\boldsymbol{p}^*) = \sum_i \left(p_i^* - c_i^*\right)^2 \leq OCV(\boldsymbol{p}) = \sum_i \left(p_i - c_i^*\right)^2$$

for all vectors of prices for winning bids ***p*** that satisfy the first, second and third requirements (where we will show below that the inequality is strict with respect to all vectors of prices for winning bids ***p*** other than the vector of winning prices ***p****).

4.8 We now proceed to prove that:

1. there exists a winning price for each winning bid that meets all four of the requirements;

2. for any given set of valid auction stage bids and winning combination of auction stage bids, the winning price for each winning bid is unique – there is only one winning price for each winning bid that meets all four of the requirements.

Proof of existence of winning prices

4.9 We first note that the vector of prices for winning bids where the price for each winning bid is equal to the amount of that winning bid (i.e. $p_i^* = b_i^*$ for all bidders *i*) always satisfies the first and second requirements. Hence there always exists at least one vector of prices for winning bids that meets both of these requirements.

4.10 Secondly we note that the third and fourth requirements identify winning prices as being the minima of two continuous functions of prices for winning bids over the subset of vectors of prices that satisfy the first and second requirements. Hence provided that the subset of vectors of prices that satisfy the first and second requirements is closed and bounded, those minima will exist, and the subsets of vectors of prices where those functions attain those minima will themselves be (non-empty, closed) subsets of the vectors of prices that satisfy the first and second requirements.

4.11 The subset of vectors of prices that satisfy the first and second requirements is clearly bounded, since the first requirement imposes an upper and a lower bound on the price for each winning bid.

4.12 That it is closed follows immediately from the fact that the inequalities in the first and second requirements admit equality in all cases.

4.13 Hence there must exist at least one vector of prices that satisfies all four of the requirements.

Proof of uniqueness of winning prices

4.14 We prove uniqueness of winning prices by considering the implications if the proposition were false i.e. if for some set of valid auction stage bids and winning combination of auction stage bids there existed at least two distinct vectors of prices for winning bids that met all four requirements. We consider a third vector of prices which is the average of the first two (i.e. the price for each winning bid is the average of the prices for that same winning bid in the first two vectors). This vector of prices must necessarily be distinct from each of the first two. We show that this third vector of prices also meets all of the first three requirements. We further show that the Opportunity Cost Variance for this third vector of prices is strictly less than the Opportunity Cost Variance for each of the first two vectors of prices. This contradicts those vectors of prices satisfying the fourth requirement. All of this analysis holding true irrespective of the details of the set of valid auction stage bids and winning combination of auction stage bids considered, the only possible conclusion is that there can not in fact be two distinct vectors of prices that meet all four requirements; that is to say for every set of valid auction stage bids and winning combination of auction stage bids there exists only one price for each winning bid that meets all four of the requirements.

4.15 To proceed, let $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ be two distinct vectors of prices for winning bids (i.e. $\boldsymbol{p}_1 \neq \boldsymbol{p}_2$) that each satisfy all four of the requirements for winning prices. Let $\overline{\boldsymbol{p}}$ be the average of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, that is to say $\overline{p}_i = \tfrac{1}{2}\left(p_{1i} + p_{2i}\right)$ for each bidder i.

4.16 It is then immediately clear that $\overline{\boldsymbol{p}}$ satisfies the first requirement of winning prices since:

$\overline{p}_i = \tfrac{1}{2}\left(p_{1i} + p_{2i}\right) = \tfrac{1}{2} p_{1i} + \tfrac{1}{2} p_{2i} \geq \tfrac{1}{2} r_i^* + \tfrac{1}{2} r_i^* = r_i^*$ for all bidders i; and

$\overline{p}_i = \tfrac{1}{2}\left(p_{1i} + p_{2i}\right) = \tfrac{1}{2} p_{1i} + \tfrac{1}{2} p_{2i} \leq \tfrac{1}{2} b_i^* + \tfrac{1}{2} b_i^* = b_i^*$ for all bidders i.

4.17 So far as the second requirement is concerned, we first observe that for any vector of prices for winning bids $\boldsymbol{p}$:

$$\sum_i \sum_j x_{ij}^*.\left(b_{ij} - \left(b_i^* - p_i\right)\right) = \sum_i \sum_j \left(x_{ij}^*.b_{ij} - x_{ij}^*.b_i^* + x_{ij}^*.p_i\right) = \sum_i p_i \ .$$

Hence a vector of prices $\boldsymbol{p}$ satisfies the second requirement if and only if:

$\sum_i p_i \geq \sum_i \sum_j x_{ij}.\left(b_{ij} - \left(b_i^* - p_i\right)\right)$ for all valid combinations of bids $\boldsymbol{x}$,

i.e. if and only if:

$\sum_i \sum_j \left(1 - x_{ij}\right).p_i \geq \sum_i \sum_j x_{ij}.\left(b_{ij} - b_i^*\right)$ for all valid combinations of bids $\boldsymbol{x}$.

4.18 In the case of the vector of prices $\overline{\boldsymbol{p}}$:

$$\begin{aligned}
\sum_i \sum_j \left(1 - x_{ij}\right).\overline{p}_i &= \sum_i \sum_j \left(1 - x_{ij}\right).\tfrac{1}{2}.\left(p_{1i} + p_{2i}\right) \\
&= \tfrac{1}{2} \sum_i \sum_j \left(1 - x_{ij}\right).p_{1i} + \tfrac{1}{2} \sum_i \sum_j \left(1 - x_{ij}\right).p_{2i} \\
&\geq \sum_i \sum_j x_{ij}.\left(b_{ij} - b_i^*\right)
\end{aligned}$$

for all valid combinations of bids $\boldsymbol{x}$, since each of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ satisfies the second requirement. Hence $\overline{\boldsymbol{p}}$ satisfies the second requirement.

4.19 That $\overline{\boldsymbol{p}}$ satisfies the third requirement follows trivially from the fact that:

$$\sum_i \overline{p}_i = \sum_i \tfrac{1}{2}\left(p_{1i} + p_{2i}\right) = \tfrac{1}{2} \sum_i p_{1i} + \tfrac{1}{2} \sum_i p_{2i}$$

and that each of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ satisfies the third requirement.

4.20 Finally, considering the Opportunity Cost Variance of $\bar{\boldsymbol{p}}$:

$$
\begin{aligned}
OCV(\bar{\boldsymbol{p}}) &= \sum_i \left(\bar{p}_i - c_i^*\right)^2 = \sum_i \left(\tfrac{1}{2}(p_{1i} + p_{2i}) - c_i^*\right)^2 \\
&= \sum_i \left(\tfrac{1}{2}\left(p_{1i} - c_i^*\right) + \tfrac{1}{2}\left(p_{2i} - c_i^*\right)\right)^2 \\
&= \tfrac{1}{4}\sum_i \left(\left(p_{1i} - c_i^*\right)^2 + \left(p_{2i} - c_i^*\right)^2 + 2\left(p_{1i} - c_i^*\right)\left(p_{2i} - c_i^*\right)\right) \\
&= \tfrac{1}{4}\sum_i \left(2\left(p_{1i} - c_i^*\right)^2 + 2\left(p_{2i} - c_i^*\right)^2 - \left(\left(p_{1i} - c_i^*\right) - \left(p_{2i} - c_i^*\right)\right)^2\right) \\
&= \tfrac{1}{2}\sum_i \left(p_{1i} - c_i^*\right)^2 + \tfrac{1}{2}\sum_i \left(p_{2i} - c_i^*\right)^2 - \tfrac{1}{4}\sum_i \left(p_{1i} - p_{2i}\right)^2 \\
&= \tfrac{1}{2}OCV(\boldsymbol{p}_1) + \tfrac{1}{2}OCV(\boldsymbol{p}_2) - \tfrac{1}{4}\sum_i \left(p_{1i} - p_{2i}\right)^2
\end{aligned}
$$

4.21 But $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are both required to satisfy the fourth requirement and hence their Opportunity Cost Variances must be the same i.e. $OCV(\boldsymbol{p}_1) = OCV(\boldsymbol{p}_2) = OCV^*$ say. In which case $\tfrac{1}{2}OCV(\boldsymbol{p}_1) + \tfrac{1}{2}OCV(\boldsymbol{p}_2) = OCV^*$.

4.22 But in that case:

$$
\begin{aligned}
OCV(\bar{\boldsymbol{p}}) &= \tfrac{1}{2}OCV(\boldsymbol{p}_1) + \tfrac{1}{2}OCV(\boldsymbol{p}_2) - \tfrac{1}{4}\sum_i \left(p_{1i} - p_{2i}\right)^2 \\
&= OCV^* - \tfrac{1}{4}\sum_i \left(p_{1i} - p_{2i}\right)^2 \\
&< OCV^* = OCV(\boldsymbol{p}_1) = OCV(\boldsymbol{p}_2)
\end{aligned}
$$

since $\tfrac{1}{4}\sum_i \left(p_{1i} - p_{2i}\right)^2 > 0$ (since for at least one winning bid $p_{1i} \neq p_{2i}$). This contradicts each of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ satisfying the fourth requirement.

4.23 Hence there can not exist two distinct vectors of prices that both satisfy all four of the requirements, there can only be at most one.

Section 5

Further information

5.1 Further information is available from the Ofcom auction team at spectrumawards@ofcom.org.uk.

5.2 Ofcom has dedicated a portion of its website to the award and this can be found at http://www.ofcom.org.uk/radiocomms/spectrumawards/liveawards/award_1452/