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Annex A Angle of Arrival Estimation Methods

A.1 Introduction

There are a number of different methods of estimating the angles of arrivals (AOA) of multiple plane waves using an antenna array. Throughout this section it is assumed that the array is a uniform linear array (ULA), however, the techniques are applicable to other arrays, such as circular arrays, if the array response vector is known\textsuperscript{16}. The methods are broadly categorised into (i) Fourier-based methods, which obtain the steered response of the antenna, (ii) parametric methods, which attempt to match the signals received on each antenna element to a particular far-field spatial distribution via a model of the far-field distribution and (iii) sub-space decomposition methods, such as Pisarenko’s method, MUSIC and ESPRIT and their numerous variants. This Annex provides a comparison of the main methods of estimating the AOA of multiple arriving signals. The performance of the various methods is compared using Matlab simulations that were carried out as part of this workpackage under a consistent set of conditions. Further information on these methods can be found in [1,2].

A.2 Signal Model

Throughout this Appendix, it is assumed that the signal received by the antenna array comprises multiple, \( K \), modulated sinusoidal carriers that each arrive from different angles of arrival. The \( i^{th} \) signal has the form:

\[
s_i(t) = A_i(t) \cos(2\pi f_{c_i} t + \alpha_i) \quad \text{for } 1, 2, \ldots, K
\]  

(A.1)

where \( A_i(t) \in \pm 1 \) is the data modulation (assumed to be BPSK for simplicity), \( f_{c_i} \) is the carrier frequency and \( \alpha_i \) is the random phase of the carrier when it arrives at the reference element of the antenna array. The frequencies of each carrier may be the same, \( f_{c_i} \), or they may be different, either due to the effect of different Doppler shifts at the different angles of arrival, or because they represent different source frequencies.

\begin{footnote}
\textsuperscript{16} The array response vector is sometimes known as the array manifold. It relates the electrical phase shift of the incoming (plane) wave front of the signal at each of the elements in the array due to the path length difference between the wave front and the array element. As such it relates the spacing and geometry of the array elements with the angle of arrival of the plane wave.
\end{footnote}
For the most part, it will be assumed that the modulating signal is different for each user so that the various signals, although co-channel are uncorrelated. In some cases it will be assumed that the arriving signals are coherent. In this case, the data modulation and the carrier frequency is assumed to be identical for all signals at the different angles of arrival; however, the phase of each signal on arrival at the antenna is assumed to be different. This condition corresponds to the case of multipath reception from a single static source.

The output of the array, expressed as a vector, is given by:

\[ x(t) = s(t)V + n(t) \]  \hspace{1cm} (A.2)

where

\[ V = \begin{bmatrix} v_1 \\ \vdots \\ v_K \end{bmatrix} \]

and \( x(t) \) is the \((1\times M)\) vector of signals on the \( M \) elements of the array for the signal samples at time \( t \), \( s(t) \) is the \((1\times K)\) vector of the signal samples at time \( t \), \( V \) is the \((K\times M)\) array response matrix (where the \( i \)th row corresponds to the \((1\times M)\) array response vector \( v_i(t) \) for the \( i \)th signal of frequency \( f_i \) at an azimuth angle to the array of \( \phi_i \)), and \( n(t) \) is the \((1\times M)\) vector of additive Gaussian noise samples at each element at time \( t \). The additive noise may represent the normal additive noise processes found in all receivers, or it may additionally represent a spatially random signal due to interference.

**A.3 Fourier-based methods**

Fourier-based methods [3,4,5] are the most robust method of AOA estimation, since they require no a priori information regarding the number, \( L \), of co-channel signals that may impinge on the array (subject generally to the limit, \( L \leq M - 1 \), where \( M \) is the number of elements in the array).

Consider a plane wave arriving at an azimuth angle \( \phi \) to the ULA, as shown in figure A.1.

---

**Figure A-1 : Geometry used for the uniform linear array**
The wavefront at two adjacent elements has a path length difference of \( d \sin \phi / c \), and this results in an electrical phase shift of \( \theta = 2\pi d \sin \phi / \lambda \) between the two elements, where \( \lambda \) is the wavelength of the signal and \( d \) the spacing between antenna elements. The phase shift of the signal at the \( m^{th} \) element relative to the first element (labelled 1 in figure A.1) is [1]:

\[
\theta_m = (m - 1) \frac{2\pi d \sin \phi}{\lambda} \quad (A.3)
\]

and the corresponding array response vector for the ULA is:

\[
v(\phi) = \begin{bmatrix} e^{-j2\pi d \sin \phi / \lambda} & \ldots & e^{-j(M-1)2\pi d \sin \phi / \lambda} \end{bmatrix}^T \quad (A.4)
\]

where \([\cdot]^T\) represent transposition of the vector.

The antenna array can be seen as a method of spatially sampling wavefronts propagating at a particular carrier frequency. For the ULA, the ‘spatial sampling frequency’, \( U_s \), is given by:

\[
U_s = \frac{1}{d} \quad (A.5)
\]

With temporal signals, the phase progression for uniform sampling is a consequence of the frequency; that is, consecutive unit time samples of the same signal differ only by a phase shift \( e^{j2\pi F} \), where \( F \) is the frequency. For the case of a spatially propagating signal the phase progression is due to the distance propagated by the wavefront and this frequency is represented by the spatial frequency:

\[
U = \frac{\sin \phi}{\lambda} \quad (A.6)
\]

and the normalised spatial frequency is:

\[
u = \frac{d \sin \phi}{\lambda} \quad (A.7)
\]

Consequently, the array response vector can be written in terms of the normalised spatial frequency as:

\[
v(\phi) = \begin{bmatrix} e^{-j2\pi u} & \ldots & e^{-j(M-1)2\pi u} \end{bmatrix} \quad (A.8)
\]

The inter-element spacing is the spatial sampling interval and similar to Shannon’s theorem for discrete-time signals, the normalised frequencies are unambiguous over the range \(-\frac{1}{2} \leq u \leq \frac{1}{2} \). Since the full range of possible unambiguous angles that can be detected by the ULA is \(-90^\circ \leq \phi \leq 90^\circ \), this requires that:
\[ d \leq \frac{\lambda}{2} \quad (A.9) \]

to prevent ambiguities in angle measurement over the full range. This analogy between time and frequency of temporal signals and spatial sampling and spatial frequency, explains how Fourier methods can be used to obtain the AOA of the signal.

The output of the array is the weighted sum of the signals received at each element:

\[ y_n = \sum_{m=1}^{M} c_m x_m(n) = c^H x(n) \quad (A.10) \]

where * means conjugation, \( H \) means the Hermetian operation (conjugate transpose), \( x_m(n) \) is the \( n^{th} \) sample of the sampled signal obtained at the \( m^{th} \) array element and \( c_m \) is the complex valued weighting coefficient of the \( m^{th} \) antenna element. Also, \( c \) is the vector of the weight coefficients and \( x(n) \) is the vector of the \( n^{th} \) signal sample received at each element.

The array acts as a spatial matched filter when the weight coefficients are set equal the array response vector for a signal whose wavefront has an azimuth angle \( \phi_s \) to the array:

\[ c_{mf}(\phi_s) = v(\phi_s) \quad (A.11) \]

since in this case, the phase delay inserted by the spatial separation of the elements is perfectly compensated by the phase advance provided by the weights (hence the need for the conjugation operation in (A.10)). Consequently, the scalar output of the spatial matched filter is given by:

\[ y_n(\phi_s) = c_{mf}^H(\phi_s)x(n) = v^H(\phi_s)x(n) \quad (A.12) \]

It will be seen from (A.12), given the form of the steering vector given by (A.8), that the steered response of the spatial matched filter has precisely the same form as the discrete-time Fourier transform, in which spatial frequency replaces temporal frequency. This function is continuous in spatial frequency.

Suppose that we do not know the direction of the signal. One method is to repeatedly change the look direction, \( \phi \), used for the array response vector and for each value of \( \phi \) obtain the corresponding output of the array for that direction using (A.12). This is called the steered response of the array. In this case, the use of \( v(\phi) \) in this way is referred to as a steering vector as it is used to point the array (via the array coefficients \( c \)) in a particular look direction. The maximum of the absolute value of the steered response will correspond to the situation where the look direction of the steering vector points to the true angle \( \phi_s \) of the wanted signal. An alternative form that can be used for AOA estimation is to use the spatial power response instead of the absolute value of the steered response, which is given by:

\[ C(\phi) = |y_n(\phi_s)|^2 = |c_{mf}^H(\phi_s)x(n)|^2 = |v^H(\phi_s)x(n)|^2 \quad (A.13) \]
Since (A.12) and (A.13) are continuous in $\phi$, the grid size used to select the various $\phi$ values can be extremely fine, although this is at the expense of computational effort due to the calculation of a very large number of complex exponentials. A less computationally intensive approach is to compute the steered response vector at a finite number of discrete values of $\phi$ using the discrete Fourier transform in place of the discrete-time Fourier transform to obtain the spatial power response in terms of the spatial frequency, $u$:

$$C(u) = |\mathcal{F}(x(n))|^2$$  \hspace{1cm} (A.14)

and then to convert the spatial frequency to angle, $\phi$, via:

$$\phi = \sin^{-1}\left(\frac{\lambda u}{d}\right)$$  \hspace{1cm} (A.7)

However, consider the situation where the array has only eight elements. By directly applying the discrete Fourier transform to the eight sample values, $x(n)$, provides only eight corresponding points in both the spatial spectrum and the azimuth angle over the range $\pm 90^\circ$. This is far too coarse to pinpoint the required AOA. The solution is to use the zero-padded FFT. This corresponds to the situation where the real array elements containing the signal samples $x(n)$ are supplemented by many additional dummy elements whose signal samples are taken as zero. Consequently, 8 real sample values could be supplemented by 1016 dummy points (say) and the spatial power response computed using a 1024 point FFT algorithm giving a much finer spatial frequency response curve.

However, for realistically sized arrays, zero-padding does not improve the angular resolution of the antenna array due to the fact that, because of the finite number of array elements, the steered response of the spatial matched filter beamformer produces a finite width main-lobe as well as sidelobes. Furthermore, because of the non-linear relationship between the spatial frequency and azimuth angle, the sidelobe structure is heavily distorted in terms of angle. The effect of the broad main lobe on accurate AOA estimation is felt particularly if the signal snapshot, $x(n)$, is corrupted by noise, since the precise peak of the mainlobe, indicating the AOA, is determined by the noise corrupting that particular snapshot.

Figure A.2a shows the normalised steered response of an 8 element array to a single far-field source at an angle, $\phi = 25^\circ$ to the array, using the geometry of figure A.1. In this case the frequency is chosen to be 1.5GHz and the antenna spacing at this frequency is chosen to be $\lambda/2$ (overall length of array, 1.6m). The signal is received in the absence of noise and in order to obtain this response, a 1024-point zero-padded FFT is used (i.e. 1016 inserted zeros). Note the beamwidth of the mainlobe and the sidelobes.

---

17 In practice, a fast Fourier transform algorithm would be used in place of the discrete Fourier transform as it is computationally less intensive but this places a restriction that the number of sample points used (including dummy zeros, if zero padded) is $N=2^n$, where $n$ is an integer.
Contrast this with the normalised steered response of a much larger, 32 element array (overall length of array 6.4m), as shown in figure A.2b, where, as expected, the width of the main lobe is much reduced.

![Diagram](image1.png)

(a) 8 Element ULA  
(b) 32 Element ULA

**Figure A-2 : Steered response of an a linear array to a far field source at 25° using FFT method**

Furthermore, even in the absence of noise, it may be very difficult to accurately resolve two or more signals arriving at different AOAs if they fall within the main lobe, particularly if one signal is much stronger than the other since the steered response is the convolution of the spatial matched filter beam pattern with the spatially separated sources, and the peak in the spatial power response may not correspond to the AOA of any of the signals! This problem is not confined to sources located within the main lobe. It is quite possible that a strong signal well separated from a weaker signal may produce sidelobes in the spatial power response that heavily distort the mainlobe peak of the weaker source, making it difficult to pinpoint the precise AOA. Figure A.3a and A.3b show the ability of the Fourier method to resolve two far-field sources at 25° and 35° in the absence of noise. A 1024 point zero-padded FFT is used to obtain this response. In this case it should be noted that the two signals are coherent (i.e. the data modulation for both signals is assumed to be identical and the two carrier frequencies are identical).

![Diagram](image2.png)

(a) 8 Element array  
(b) 32 Element array

**Figure A-3 : Steered response of ULA for two far field sources at 25° and 35° using FFT method**
It is possible to suppress the effect of the sidelobes, generally at the expense of further broadening the width of the main lobe, by using appropriate taper functions [6]. These taper functions, $t$, weight the signal sample snapshot, $x(n)$ in amplitude prior to the signals being weighted in phase by the beamformer coefficients $c$, i.e. the effective weights become $c_t = c \otimes t$, where $\otimes$ represents element by element multiplication. These amplitude weights are generally symmetrical about the central element(s) of the array with a maximum in the centre, tapering to a smaller value at each end of the array. The shape of the taper is generally smooth, such as a Gaussian-like shape or a raised cosine. There are many different types of taper function available, each providing different levels of sidelobe suppression and main-lobe broadening. A common taper is the Dolph-Chebyshev taper:

$$t(n) = \begin{cases} C \cos\left(\frac{(N-1)\cos^{-1}\left(\frac{\beta \cos n}{2}\right)}{2}\right) & \text{for } \beta \cos n / 2 \leq 1 \\ C \cosh\left(\frac{(N-1)\cosh^{-1}\left(\frac{\beta \cos n}{2}\right)}{2}\right) & \text{for } \beta \cos n / 2 > 1 \end{cases}$$

(A.15)

where $\beta = \cosh\left(\frac{(N-1)\cosh^{-1}\left(10^{-0.05\alpha}\right)}{2}\right)$ and $\alpha$ is the sidelobe level needed in dB. This taper provides good general performance, but others include Hanning tapers, Hamming tapers and Kaiser tapers. A less good feature of the taper is that it reduces the spatial response of the beamformer to the wanted signal(s). Figure A.4 shows the impact of using the Dolph-Chebyshev taper with $\alpha=-3$dB for the case of figure A.3a when the 8 element array was unable to resolve the two signal sources. Ironically, although this has appeared to increase the sidelobe level, it has improved resolution.

![Figure A-4: Use of the Dolph-Chebyshev taper to help resolve the two far-field sources at 25° and 35°, which were not resolvable with the untapered 8 element ULA shown in figure A.3a](image)

Invariably, the signal of interest will be corrupted by noise, and figure A.5a shows the effect of AWGN (receiver noise or clutter) at an SNR=10dB on the steered response for the 8 element ULA for a single far-field source at 25°. Shown in this figure are 10 snapshots of the steered response.
Figure A-5: Impact of a 10dB SNR on the steered response of the 8 element ULA to a single source at 25°
(a) raw snapshots (b) averaged periodogram response after averaging 100 snapshots

Figure A.6 shows how the effect of noise impacts on the estimate of the AOA. In this set of figures, the angle of arrival is taken as the angle corresponding to the maximum in the steered response. In this case 2000 snapshots of the steered response for a single source at an azimuth angle of 25° were taken and a histogram of the indicated angle of arrival taken for SNRs of (i) 10dB, (ii) 0dB and (iii) -7dB. (Note the x axis scale is different for each figure).
One way of reducing the effect of the signal noise on the accuracy of the AOA spectrum is to use $P$ snapshots of the array signal samples, $x_1(n), \ldots, x_P(n)$, and to use the averaged periodogram to obtain the average spatial power response $\overline{C}(\phi)$ by performing $P$ spatial power responses ($C_p(\phi)$) and to take the average of each spectral coefficient from the $P$ values.

$$\overline{C}(\phi) = \frac{1}{P} \sum_{p=1}^{P} C_p(\phi)$$  \hspace{1cm} (A.16)

This approach is computationally intensive as it relies on taking $P$ zero-padded FFTs of size $N$ bins and then performing $N$ averages, each containing $P$ values. Nevertheless, it is effective, as figure A.5b shows for the case where $P=100$ snapshots were used in the averaging process. In this case, to the precision of a 1024 point FFT, even after 1000 tests, there was no error in estimating the correct AOA at an SNR=-7dB or better.

An alternative approach, which is generally less computationally intensive than the averaged periodogram, is to perform averaging first and then to take a single zero-padded FFT. In this case, the averaging takes the form of the spatial correlation of the signal samples across the antenna array. Spatial correlation represents how the average signal phase changes from one antenna element to the next. This information is provided by the spatial correlation function described in the next section.

### A.4 Spatial Correlation

The samples of the signal at the individual elements represent a ‘snapshot’ of the variation of the phase across the array and each snapshot can be used to obtain the instantaneous AOA for those samples, as described above. However, since each signal sample is noisy the accuracy of the AOA estimate is degraded, since the noise samples transform into a noisy spatial power response. The array correlation matrix of the signal vector $\mathbf{x}(n)$ shows the individual correlations of the received signals between all the elements of the array.

$$\mathbf{R}_\mathbf{x} = E[\mathbf{x}\mathbf{x}^H]$$  \hspace{1cm} (A.17)
where $E[ ]$ is the expectation operation. In this expression reference to the sample number $n$ has been removed because the expectation is obtained from the long term average of a great many signal vector snapshots. The array correlation matrix has the form:

$$
R = 
\begin{bmatrix}
E[x_1^2] & E[x_1x_2^*] & \cdots & E[x_1x_M^*] \\
E[x_2x_1^*] & E[x_2^2] & \cdots & E[x_2x_M^*] \\
\vdots & \vdots & \ddots & \vdots \\
E[x_Mx_1^*] & E[x_Mx_2^*] & \cdots & E[x_M^2]
\end{bmatrix}
$$

(A.18)

In practice, the expectation of each element of $R$ is approximated by average values obtained over an acceptably large number of samples, $P$, commensurate with the stationarity of the signal statistics:

$$
E[x_i x_j] \approx \frac{1}{P} \sum_{n=1}^{P} x_i(n)x_j^*(n)
$$

(A.19)

In practice the two operations can be conveniently combined by storing the signal vector for $P$ snapshots as a matrix, $X$:

$$
X =
\begin{bmatrix}
x_1(n) & x_2(n) & \cdots & x_M(n) \\
x_1(n+1) & x_1(n+1) & \cdots & x_1(n+1) \\
\vdots & \vdots & \ddots & \vdots \\
x_1(n+p) & x_1(n+p) & \cdots & x_1(n+p) \\
\vdots & \vdots & \ddots & \vdots \\
x_1(n+p) & x_1(n+p) & \cdots & x_1(n+p)
\end{bmatrix}
$$

(A.20)

such that:

$$
\hat{R} = \frac{1}{P} X^H X
$$

(A.21)

This is a very straightforward operation. The hat signifies that the correlation matrix is an estimate rather than the true correlation matrix. However, it is important to recognise that the error in the estimate of $\hat{R}$, impacts on the accuracy of all the methods of spectrum estimation that rely on it.

Note that the correlation matrix is different from the correlation function since the latter obtains the overall correlation at a particular ‘lag’. So for example, the first term of the correlation function would comprise:

$$
\hat{R}(0) = \frac{1}{M} \sum_{i=1}^{M} E[|x_i|^2]
$$

(A.22)

the second term would comprise:
\[ \hat{R}(n) = \frac{1}{M} \sum_{i=1}^{M-1} E[x_i x_{i+n}^*] \]  
(A.23)

and for an arbitrary lag, \( k \)
\[ \hat{R}(k) = \frac{1}{M} \sum_{i=1}^{M-k} E[x_i x_{i+k}] \]  
(A.24)

However, it should be recognised that the correlation function is symmetrical about zero lag and defined over the range of lags from \( \pm M \). The correlation function may also be obtained in an approximate form by replacing the expectations by simple averages.

The loss of correlation along the array due to the change in phase of the signal provides all the information needed to obtain the average spatial power response. This is achieved by taking the zero-padded FFT of the correlation function:
\[ \overline{C}(u) = \sum_{m=-M}^{M} \hat{R}(m)e^{-jum} \]  
(A.25)

and converting the average spatial frequency spectrum into average spatial power response using:
\[ \phi = \sin^{-1}\left(\frac{\lambda u}{d}\right) \]  
(A.26)

to convert from frequency to angle. The above method is sometimes called the correlogram. However, it still has the same resolution as the periodogram, which is limited by the number of array elements.

### A.4.1 Comment on using multiple snapshots – effect of signal bandwidth

It is essential that the snapshots used to form the correlation matrix are statistically independent. If it is assumed that the multistatic radar receivers have a bandwidth of \( B \), the minimum time between successive snapshots should be \( \tau_s > 1/B \). For example if the receiver has a bandwidth of 25kHz, the minimum sampling time should be 40\( \mu \)s. This means that if it is necessary to take 1000 snapshots to provide sufficient resolution to the AOA estimate, this can be achieved in 40ms. For a platform travelling at 200mph, this equates to a distance error of 3.4 metres whilst the samples are collected. However, by increasing the receiver bandwidth to 100kHz, the time take to acquire 1000 samples reduces to 10ms and the platform positional error reduces to 83cm. In practice, the estimated position of the platform will dependent upon the geolocation algorithm and the impact of geometric dilution of precision, and the positional error due to the aperture time of the AOA algorithm can be viewed as the minimum spatial error.

### A.5 Minimum Variance Method

In terms of spectral estimation, the minimum variance estimator, also known as the Capon beamformer [7], generally outperforms the periodogram and the correlogram method,
described in the previous section and it has the advantage that it uses the correlation matrix directly rather than the correlation sequence. It is often referred to as a high resolution spectral estimator and it also has the advantage that it is non-parametric, which means that it does not assume an underlying model, whose parameters need to be known a priori.

The minimum variance spectral estimate can be shown to be:

$$
\hat{R}_{mv}(\phi) = \frac{M}{v^H(\phi)R^{-1}v(\phi)}
$$

(A.27)

where $v(\phi)$ is the steering vector at some azimuth angle $\phi$. This means that the spectrum $\hat{R}_{mv}(\phi)$ has to be obtained by forcing the look angle $\phi$ through all angles of interest. Figure A.7 shows the equivalent steered response for the same 8 element array as for the previous examples for a single far-field source at an azimuth angle of 25° but using the minimum variance spectral estimator. As for the previous examples, the number of look angles is set at 1024, distributed over ±90°. It should be noted that the SNR of the signal at each element is 0dB prior to signal averaging. However, the estimated correlation matrix in this figure is produced by averaging over 500 signal vector snapshots.

Figure A-7: Equivalent steered response of an 8 element array to a single source at 25° using the minimum variance method at an SNR=0dB (500 snapshot averaging)

It is immediately clear why this is referred to as a high resolution method. Figure A.8a shows the same steered response estimator for three coherent signal sources located at -35°, +25° and +40° for the case where each signal is equal power and the SNR for each signal is 0dB. Although it appears from this figure that this estimator ought to able to resolve better than 15°, this is virtually the limit at this SNR. Shown in figure A.8b, is the situation where the three signals were not coherent (due to random data on each source. Again, 15° appears to be the limit of the resolution of this method for an 8 element array at an SNR=0dB per source, but with 500 snapshot averaging to form $\hat{R}$. However, if the number of elements is increased to 32, then the resolution is significantly improved, as shown in figure A.9 and it is able to resolve 4° source separation with ease, and 3° with more difficulty. In this figure, the sources are located at -35°, +25° and +29°. As before,
case (a) is for coherent sources and (b) is for non-coherent sources. Although the resolution is no higher for case (b) than case (a), the determination of the three AOAs seems to be much clearer for the non-coherent case.

Figure A-8: Steered response of an 8 element array using the minimum variance method for signal sources at -35°, 25° and 40° at an SNR=0dB per signal source (500 snapshot averaging)

(a) Coherent Sources
(b) Non-coherent sources

Figure A-9: Steered response of a 32 element array using the minimum variance method for signal sources at -35°, 25° and 29° at an SNR=0dB per signal source (500 snapshot averaging)

(a) Coherent sources
(b) Non-coherent sources

The major penalty of the minimum variance method is the need for significant averaging to produce the estimate of the correlation matrix $\hat{R}$. If this is not carried out, and the estimate of $\hat{R}$ is poor, the estimate of the steered response is also extremely poor. As an example, figure A.10a shows the effect of using only 10 snapshots of the signal vector $x$ even though the SNR is 10dB. The array had 8 elements and the three sources were located at -35°, 25° and 40°. Coherent sources were used to obtain this result. In this case, one wanted source is not detected and additional peaks which are extremely dependent on the noise, are observed. Figure A.10b shows the same case after 50 snapshots are used to form $\hat{R}$ and figure A.10c after 100 snapshots are used.
A.6 Super-resolution methods based on Eigen-decomposition

Although Fourier-based methods are robust, the relatively low number of array elements that can be accommodated in practical arrays means that the broad main-lobe beamwidth of the steered response is a major limiting factor to the resolution of Fourier methods. When using array processing to obtain the AOA, we are attempting to identify the spatial frequency of the complex exponentials representing the phase shift of the wavefronts of the various signals as they propagate along the array. This is particularly true as long as the wavefront of each signal is flat when each wavefront corresponds to a single frequency complex exponential. Consequently, the method described in this section is based on a harmonic signal model and the method obtains the parameters (i.e. frequency) of the complex exponentials that form the basis of the method.

Unlike the Fourier methods where the signal is decomposed into \( N \) complex exponentials (or \( N/2 \) sinusoids/cosines) arranged on a fixed frequency interval on the basis of taking \( N \) sample values, here the signal is assumed to comprise of a limited number of complex exponentials whose frequencies must be evaluated. The advantage is that the frequency separation of these complex exponentials may be very close (much closer than the equivalent main lobe beamwidth of the Fourier methods). Consequently, these methods are often referred to as super-resolution methods. However, the success of these methods is very much dependent upon the accuracy of the underlying model. Consequently, if it is known that there are four signals at different AOAs, the model will find the parameters of four complex exponentials. If, in reality, there are only three signals but the model assumes four, four AOAs will be found(!), and it is not guaranteed that any of the frequencies found by the method will then be correct!

Eigen-methods of AOA determination (also known as subspace methods) also use the array correlation matrix as the starting point. However, in this class of methods, the array correlation matrix is decomposed into its eigenvalues and eigenvectors:

\[
\hat{R} = Q \Lambda Q^H
\]  

(A.28)

where \( \Lambda \) is an \((M\times M)\) diagonal matrix comprising the eigenvalues while \( Q \) is an \((M\times M)\) matrix whose columns, \( q_1, \ldots, q_M \), are the corresponding eigenvectors.
It is assumed that each of the elements of the correlation matrix consist of a wanted term representing the degree of correlation between the elements and a noise term due to the limited averaging employed to obtain the correlation. It is assumed that the noise term is uncorrelated with the wanted signal. Eigen-decomposition allows the correlation matrix to be decomposed into a matrix representing the wanted terms and a matrix representing the noise terms:

\[ \mathbf{R} = \mathbf{Q}_S \mathbf{\Lambda}_S \mathbf{Q}_S^H + \mathbf{Q}_n \mathbf{\Lambda}_n \mathbf{Q}_n^H = \mathbf{Q}_S \mathbf{\Lambda}_S \mathbf{Q}_S^H + \sigma^2 \mathbf{Q}_n \mathbf{Q}_n^H \]  

(A.30)

where \( \sigma \) is the standard deviation of the noise.

These two sub matrices are called the wanted signal subspace and the noise subspace. It is important to recognise that the signal eigenvectors are orthogonal to the noise eigenvectors. Here, \( \mathbf{\Lambda}_S \) represents a \((K \times K)\) diagonal matrix of the \(K\) largest eigenvalues and \( \mathbf{\Lambda}_n \) represents a \(((M-K) \times (M-K))\) diagonal matrix of the \(M-K\) smallest eigenvalues that are assumed to represent the noise in the spatial correlation matrix. \( \mathbf{Q}_S = [\mathbf{q}_1 \cdots \mathbf{q}_K] \) is an \((M \times K)\) matrix of the \(K\) signal eigenvectors whose columns are re-ordered to correspond with the reordering of the eigenvalues when they were ranked. \( \mathbf{Q}_n = [\mathbf{q}_{K+1} \cdots \mathbf{q}_M] \) are the \((M-K)\) noise eigenvectors corresponding to the \(M-K\) eigenvalues and, again, these are reordered to match the eigenvalue order during ranking.

\( K \) can take any value from 1 to \(M-1\). For example, if \(K=1\), it is assumed that there is only one wanted signal component whose AOA needs to be determined. The remaining \((M-1)\) terms are all assumed to be noise terms. If \(K=3\), it is assumed that there are three signals of interest all at different angles of arrival, and there are \((M-3)\) noise terms. If \(K = M-1\), then it is assumed that we are interested in finding all the \(M-1\) wanted signals whose AOAs can be found from an \(M\) element array. In this last case, there is only one noise term.

To explain the eigen-decomposition method further, assume that the \((M \times M)\) correlation matrix has been decomposed into an \((M \times M)\) diagonal matrix of eigenvalues and an \((1 \times M)\) eigenvector. The individual eigenvalues, \([\lambda_{1,1}, \lambda_{2,2}, \cdots, \lambda_{M,M}]\), are then ranked into the \(K\) largest values and the \((M-K)\) smallest values. A \((K \times K)\) diagonal matrix of the ordered largest eigenvalues and an \(((M-K) \times (M-K))\) diagonal matrix of the ordered smallest eigenvalues are created. Simultaneously, the \(K\) eigenvectors corresponding to the \(K\) largest ranked eigenvalues form the wanted set of eigenvectors and the \((M-K)\) eigenvectors corresponding to the smallest eigenvalues form the set of noise eigenvectors. From this point, there are a number of different algorithms that may be deployed, and these will be described briefly in the following sections.

Note: Implicit in the use of eigen-decomposition methods is that the user knows, \textit{a priori}, how many signals (and hence AOAs) are of interest. This is in contrast to Fourier methods where such \textit{a priori} knowledge was not required (subject to the \((M-1)\) resolvable limit). However, with eigen-decomposition methods it is not permissible to choose \(K\) to be a large ‘catch-all’ value, since the methods will always find \(K\) angles of arrival, even where only one actual signal was present because in this case, \((K-1)\) noise values have been incorporated into the wanted signal term. This means that the \((K-1)\) erroneous AOAs will be random. The
problem of correctly identifying the true number of wanted signals, $K$, is extremely complex and this still largely remains an unsolved problem.

A.6.1 Pisarenko's method

In Pisarenko’s method [8], it is assumed at the outset that there are $(M-1)$ wanted eigenvectors and one noise eigenvector:

$$Q_n = q_M$$

(A.31)

corresponding to the smallest eigenvalue, $\lambda_M$. This eigenvector must be orthogonal to the $(M-1)$ wanted eigenvectors and (without proof):

$$\nu^H(u)q_M = \sum_{k=1}^{M-1} q_M(k)e^{-j2\pi u(k-1)} = 0 \quad \text{for } m \leq M - 1$$

(A.32)

where $\nu$ is the steering vector of the array at some look direction given by $\phi = \sin^{-1}(\lambda u/d)$ and $q_M(k)$ is the $k^{th}$ element of the $M^{th}$ eigenvector. It will be immediately recognised from the RHS of (A.32) that $\nu^H(u)q_M$ represents a complex valued ‘spectrum’ whilst

$$\left|\nu^H(u)q_M\right|^2$$

is the corresponding power spectrum. Also note that at each of the $M-1$ wanted signal eigenvalues, the ‘spectrum’ falls to zero. Consequently, the Pisarenko pseudo-spectrum is defined as

$$S_{pis}(u) = \frac{1}{\left|\nu^H(u)q_M\right|^2}$$

(A.33)

The resulting pseudo-spectrum has the property that for each value of $u$ corresponding to the AOA of one of the $M-1$ wanted signals, the pseudo spectrum has a very large peak. If there is truly no correlation between the signal subspace and the noise subspace at these $M-1$ spatial frequencies, the peak is of infinite height.

Note that the pseudo-spectrum simply provides the spatial frequency of the $M-1$ signals; there is no information provided in the maximum value of the peak and it is not possible to use this method to determine whether one signal has a larger field strength than the others (unlike Fourier methods).

Having to plot the full pseudo-spectrum in order to identify the $M-1$ frequencies of the complex exponentials is computationally wasteful and unnecessary. Equation (A.33) is the

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18 Note that $S_{pis}(u)$ is evaluated at a specific spatial frequency $u$. The full pseudo spectrum is obtained by repeating the calculation (A.33) for all values of $u$, with a suitably small spacing to ensure that the peaks are accurately located.
Fourier transform of the $M^{th}$ eigenvector. However, it is also possible to take the z transform of this eigenvector:

$$Q(z) = \sum_{k=1}^{M} q_M(k) z^{-k}$$

(A.34)

which is simply an $M^{th}$ order polynomial. The phases of the $M-1$ roots of this polynomial correspond to the spatial frequencies of the $M-1$ wanted sources and these can then be equated to the actual look angle by means of (A.7). (This presupposes that finding multiple roots is computationally less intensive than performing a very fine grid FFT).

The major disadvantage of this method is that if an eight element array is used, seven signal sources are assumed to exist and the method will locate seven AOAs. If only one signal source is known to exist, only signals from the first two antenna elements should be taken. This is intuitively correct, since in this case a simple phase interferometer has been formed which can find the AOA of a single source.

As an example of the use of Pisarenko’s method, Figure A.11 shows the pseudo spectrum for a single source at an AOA of $25^\circ$. Here, the SNR is 0dB, but the correlation matrix is formed from 500 snapshots. In order to obtain this plot, only two of the eight array elements are used.

![Plot showing the pseudo spectrum for a single source at an AOA of 25°. The SNR is 0dB and 500 snapshots are used to form $\hat{R}$.](image)

Figure A-11 : Pisarenko’s method used to obtain the AOA of a single source at an AOA=25°. The SNR of the received signal is 0dB but 500 snapshots are used to form $\hat{R}$.
Figure A-12: Histogram of the AOA estimate for 2000 trials using Pisarenko’s method for SNR=0dB and the use of 500 signal snapshots to form the correlation matrix

Figure A.12 shows the spread in angle of arrival estimate at an SNR =0dB, when 500 snapshot averaging is used to form $\mathbf{R}$. In order to obtain this histogram, 2000 trials were carried out to form the histogram. It is clear from this figure that the resolution of this method is extremely high compared with a conventional Fourier based method. Since Pisarenko’s method is outperformed by a number of different variants, the case for the estimation of the AOA of multiple signals is provided in the following sections.

A.6.2 MULTIPLE SIGNAL CLASSIFIER (MUSIC)

The MUSIC algorithm is an extension of Pisarenko’s method which also exploits the orthogonality between the signal subspace and the noise subspace [9,10,11]. In this case, an arbitrary number of wanted signal eigenvectors, $K$, up to Pisarenko’s ($M-1$), can be chosen to represent the wanted signal subspace. However, what this really means is that the noise subspace contains ($M-K$) terms rather than the single term used in Pisarenko’s method. Consequently, MUSIC allows for averaging over the noise subspace and as a result it performs better in noise than Pisarenko’s method.

Consider the situation in which $K$ signal sources are known to exist, where $K<M$. In the MUSIC algorithm there are $M-K$ noise eigenvectors each of which are orthogonal to the $K$ wanted signal eigenvectors. Thus, drawing on Pisarenko’s approach, from (A.33) each noise eigenvector must have $K$ roots corresponding to the signals in the signal subspace.

$$V^H(u)q_m = \sum_{k=1}^{M-1} q_m(k)e^{-j2\pi u(k-1)} = 0 \quad \text{for } m \leq M-1$$  \hspace{1cm} (A.35)

where $q_m$ represents the $m^{th}$ noise eigenvector (where $K<m\leq M$).

All the $M-K$ eigenvectors share the same $K$ roots. However, because each noise eigenvector is of length $M$, there are an additional $M-K$ roots that are due entirely to the noise. However, each of the noise eigenvectors will have these roots due to noise at random frequencies and a way of averaging out any spurious response in the pseudo-
spectrum due to the noise is to take the average the pseudo-spectra obtained for each of the noise eigenvectors.

\[ S_{\text{music}}(u) = \frac{1}{M} \sum_{m=K+1}^{M} |v^H(u)q_m|^2 \]  

(A.36)

This is called the MUSIC pseudo-spectrum. As an example of its performance, figure A.13 shows the MUSIC pseudo-spectrum for the case of a single source at 25° obtained using an 8 element array. As for the previous cases, SNR=0dB, but 500 snapshots are used to form \( \hat{R} \). Note that for this pseudo-spectrum, and those which follow, the y axis of the spectrum is shown as dB. In this case, the pseudo-spectrum uses 1024 point resolution. To obtain this plot, it was assumed a priori that the number of signal sources was \( K = 1 \).

Figure A.13: AOA estimation of a single source at 25° using the MUSIC algorithm

Figure A.14 shows the spread in the AOA estimate at an SNR =0dB, when 500 snapshot averaging is used to form \( \hat{R} \). In order to obtain this histogram, 2000 trials were carried out to form the histogram. It is clear from this figure that the resolution of this method, like Pisarenko’s method, is extremely high compared with a conventional Fourier-based method. It is clear that the resolution of this method when estimating the AOA of a single source is less than 0.4°, under these difficult SNR conditions.
Study into Spectrally Efficient Radar Systems in the L and S Bands

A-20

Ofcom Spectral Efficiency Scheme 2004 - 2005 (SES-2004-2)

Figure A-14: Histogram of the AOA estimate for 2000 trials using the MUSIC method for SNR=0dB and the use of 500 signal snapshots to form the correlation matrix.

Figure A.15a shows the ability of the MUSIC algorithm to estimate the AOA of three uncorrelated signal sources at -35°, 25° and 30° using an 8 element antenna array. As before, the SNR for each data-modulated signal source is 0dB with 500 snapshots used to form the correlation matrix. In this case, the carrier frequency of all three source is the same, with \( f_c = 1.5\)GHz, but the data used for all three sources is random, and the initial phase of each signal at the reference antenna is also random. It was assumed a priori that \( K=3 \). Figure A.15b shows the pseudo-spectrum for SNR=30dB per source. With a 1024 point grid of azimuth angle over the range \( \pm \pi / 2 \) and a 30dB SNR, it is possible to resolve two angles of arrival separated by approximately 1° with an 8 element array. However, with a 32 element array and 2048 point grid of the azimuth angle, it is possible to resolve two sources separated by 0.4° at 30dB SNR and 500 point averaging. As for Fourier-based methods, although the spread in the AOA estimate for a single source is extremely small in high levels of noise, implying a very high resolution, in fact, it is the ability to resolve close-in AOA’s which ultimately sets the resolution of these algorithms. It should also be noted that one of the problems when estimating multiple AOAs with any of the methods is that when two or more signals arrive at similar AOAs they tend to bias the measurements, often pulling the values in slightly from their true value.
(a) 0dB SNR per source  
(b) 30dB SNR per source

Figure A-15 : AOA estimation of three non-coherent sources at -35°, 25° and 30° using the MUSIC algorithm

The following figures show the effect of making an error in the number of signal sources. In figure A.16a, it is assumed that only one source exists, whereas in figure A.16b, six sources are assumed to exist. In both cases, the SNR per source is 30dB

Figure A-16 : Effect of making the wrong assumption regarding the number of signal sources. In this case, the actual number of sources was three at -35°, 25° and 30°.

It is clear from both figures how much reliance the MUSIC method places on accurate knowledge of the model. In case (b), the model was asked to find the AOA of six sources, which it has achieved!

A.6.2.1 Use of the MUSIC method for strongly correlated sources.

Figure A.17 shows the impact of having perfectly correlated signals at arrival angles of -35°, 25° and 30°. In this case, the SNR per source is 0dB, and 500 snapshots are used in the averaging process and it is assumed that the number of sources is known. In case (a), eight antenna elements are used whereas in (b) 32 antenna elements are used. One possible scenario for this type of signal is the case of strong specular reflections.
Study into Spectrally Efficient Radar Systems in the L and S Bands

Ofcom Spectral Efficiency Scheme 2004 - 2005 (SES-2004-2)

A.6.2.2 The Eigenvector method (ev method)

The MUSIC pseudo spectrum given by (A.36) assumes that the $M$-$K$ noise eigenvalues have the same value. When the number of snapshots of $x(n)$ taken to form the correlation matrix is large, this is a reasonable assumption. However, when only a limited number of snapshots are available (as might be the case for moving targets where the radar dwell time is limited), this is less true and this distorts the pseudo-spectrum. The ev method is a minor modification of the MUSIC method that simply weights the averaging process of (A.36) by the inverse of the noise power of each noise eigenvector. Since each eigenvalue represents the noise power of that particular eigenvector, the pseudo-spectrum of the ev method is simply:

$$S_{ev}(u) = \frac{1}{\sum_{m=K+1}^{M} \frac{1}{\lambda_m} |\nu^H(u)q_m|^2}$$  \hspace{1cm} (A.37)

where $\lambda_m$ is the eigenvalue of the $m^{th}$ eigenvector.

However, implicit in all these subspace methods is that there is negligible variation in the amplitude of the signal over the antenna array. In situations where the signal is severely faded with a coherence length that is short relative to the dimension of the antenna array,
this may not be the situation and errors occur. A solution to this problem [12], when certain assumptions about the nature of the fading environment are known, can be found by modifying the MUSIC algorithm to provide a two dimensional search of both amplitude and phase distributions across the array. These modifications have not been incorporated in the models used here.

A.6.2.3 Root MUSIC

Root MUSIC [13,14,15] is a variant of the MUSIC subspace method. It does not attempt to solve the problem of obtaining the AOA of multiple coherent signals, but reduces the computational workload of the algorithm that goes into producing a high resolution pseudo spectrum. It achieves this by representing the pseudo-spectrum, $P_{\text{music}}(\phi)$, as the frequency response of a digital filter. The root MUSIC algorithm does not provide the plot of the pseudo spectrum for all $\phi$ (which is computationally intensive) but the $K$ coefficients of the corresponding to the all-pole digital filter. Since the method avoids plotting the pseudo spectrum it removes the problems of poor resolution due to choosing a too coarse grid for the $\phi$ values.

A.6.2.4 Spatially Smoothed MUSIC

Spatially Smoothed MUSIC is one method that has been used in an attempt to resolve the problem of angular resolution for multiple coherent sources. In this method, the array of length $L$ is divided into $k$ overlapping sub-arrays, each of length $P = (L - k + 1)$, as shown in figure A.18. A correlation matrix is obtained for each sub-array. A modified correlation matrix is then formed by averaging the correlation matrices of each sub-array.

$$R_{sa} = \frac{1}{k} \sum_{i=1}^{k} R_i$$  \hspace{1cm} (A.38)

This reduces the resolution of the MUSIC algorithm to that of the sub-array size, $P$, rather than the original array size, $L$. However, the reduction in the effect of coherent signals on AOA resolution is significant.